Comparison and Renaissance of Classic Line-of-Balance and Linear Schedule Concepts for Construction Industry

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Research paper

LINE-OF-BALANCE (LOB) IS A USEFUL ANALYTICAL TOOL FOR REPETITIVE ACTIVITIES IN CONSTRUCTION PROJECTS, WHICH ALLOWS SHOWING WHICH CREW IS ASSIGNED TO WHAT REPETITIVE WORK UNIT OF AN ACTIVITY. LOB is closely related to the linear scheduling method, but possesses some challenges: It must be clarified how it counts, as previous studies displayed an apparent measurement gap at the origin, implicitly representing that LOB starts at the first unit finish. Slopes in linear scheduling and LOB are different, even though both portray a measure of progress of an activity. This paper therefore tracks evolution and current use of LOB versus linear schedules. Its contribution to the body of knowledge is threefold: First, based on a literature review, LOB is found to be rooted in Activity-on-Arrow (AOA) diagrams, which makes it event-centered, not progress-centered. Differences in representing the start and productivity between LOB and linear scheduling are reviewed and explained both mathematically and graphically. Second, different LOB concepts are extracted and assessed to facilitate comparing LOB from its original use in manufacturing against the limited application of its objective chart in the construction industry. Third, a mathematical formulation based on singularity functions is developed, which can model staggering, continuity, and interruptability scenarios. Fourth, the repetitive nature of LOB and LSM enables resource-specific equations that model the level of detail of individual crews performing individual tasks.

Keywords

Line-of-Balance; Linear Scheduling; Modeling; Singularity Functions

INTRODUCTION

Repetitive activities require deploying similar resources (e.g. crews) that finish these jobs successively, which is a common phenomenon in construction. Scheduling such projects requires a method that can properly manage "the allocation of shared resources over time to competing activities" (Yamada and Nakano, 1997, p. 1). The network-based Critical Path Method (CPM) views activities and links between them as time-dependent modeling elements (Hajdu 1997), but is limited in that it "focuses strongly on the time aspect" (Lucko, 2008, p. 711) rather than the workflow, which hinders its application to scheduling repetitive activities. Researchers have therefore studied details of activity relations in different schedule representations (Hajdu and Malyusz 2014). A specialized area of research has focused on approaches that chart both time and work: Linear and repetitive scheduling techniques. While many methods exist under a plethora of names (Harris and Ioannou, 1998), such two-dimensional work-time progress profiles can clearly express important data in which researchers are interested; starts and finishes, durations, speed (productivity) of activities, buffers, criticality, and so forth. The Line-of-Balance (LOB) is "a variation of linear scheduling methods that allows the balancing of operations such that each activity is continuously performed" (Arditi *et al.*, 2002, p. 545), which is a resource-driven scheduling technique with the "primary objective ... to determine a balanced mix of resources and synchronize their work such that they fully employed" (Ammar, 2013, p. 44). But there appear to exist differences between LOB and the slightly more well-known Linear Scheduling Method (LSM): In linear schedules, an activity is represented as one line, work starts from 0, and velocity (productivity) is calculated as the slope of the line; whereas in LOB, two lines (start and finish events) are needed to represent an activity, work starts from 1, and the slope of either of its two lines represents the delivery rate. Since LSM and LOB are related models of repetitive projects, understanding the similarities and differences of their characteristics is important. Yet in Table 1 they appear to be mismatched even in their basic geometry. Since simply comparing these definitions cannot directly explain this surprising finding, the root of such substantial differences must be explored. An approach should therefore be developed that aligns features of these two promising scheduling techniques to understand their conceptual differences, as far as they may exist, and enable a more seamless use. Recommendations for creating a unified method should be derived, which could provide an integrated, powerful tool for decision-makers in the construction industry and could lead to a renaissance of linear and repetitive scheduling.

Therefore a comprehensive literature review needs to be conducted to clarify how such differences, possibly due to only partial application, have occurred and can be resolved. This research will address four Research Objectives:

- Identifying differences between LOB and linear schedule models and their original source from the literature;
- Comparing different LOB concepts that were used in manufacturing versus construction project management;
- Developing mathematical expressions for LOB in analogy to LSM equations, which are based on singularity functions, with the capability of modeling staggering, continuous, and interruptible scenarios for work progress.
- Explore how the mathematical model can provide individualized singularity functions for specific resources, which expands the model level of detail to the crews of subcontractors, who perform the actual productive work.

LITERATURE REVIEW

"The LOB methodology considers the information of how many units must be completed on any day to achieve the programmed delivery of units" (Damci *et al.*, 2013, p. 681). According to various studies (Dolabi *et al.*, 2014; Ammar, 2013; Damci *et al.*, 2013; Hegazy, 2001; Arditi and Albulak, 1986), basic steps of LOB are: (1) Draw a unit network of the repetitive activities for a single work unit; (2) estimate the crew size for each activity; (3) establish a target rate of output (delivery units/day); and (4) derive the LOB as the number of units that must

Characteristic	LSM	LOB
Activity is represented as	One line	Two parallel lines
Work starts at	0	1
Progress rate is represented as	Slope of the line	N/A
Delivery rate is represented as	N/A	Slope of any line (parallel)

Table 1: **Basic Differences of Linear Scheduling Method and Line-of-Balance**

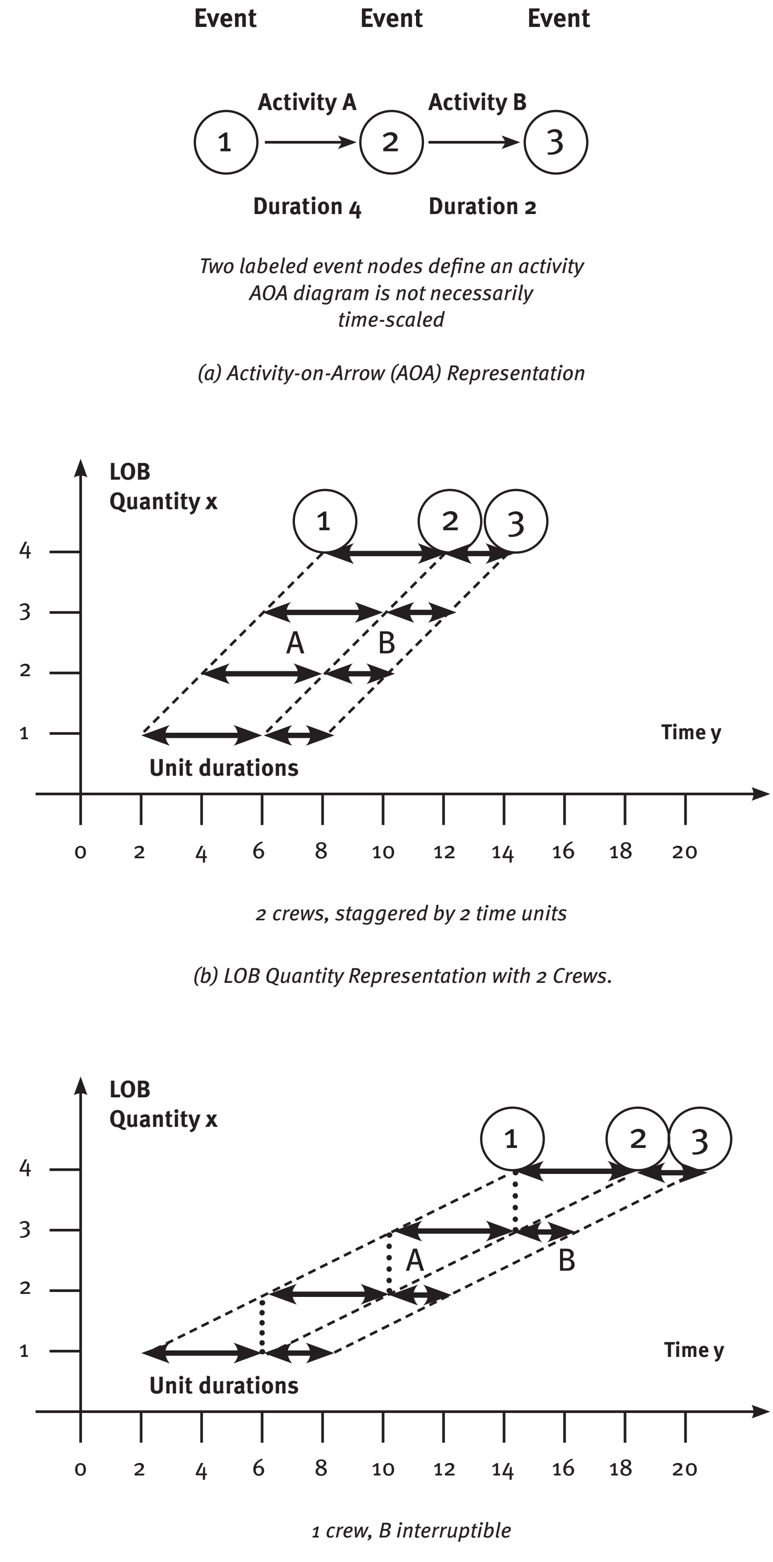
be completed at a given time. Activities in an LOB quantity chart, which first appears in Lumsden's report (1968), but not in the report by the Office of Naval Material (1962), are enveloped by two parallel lines whose slopes are the rate of output. Equation 1 models the rate of delivery m as "the slope of the line of balance joining the start times of the repetitive activity in each unit" (Arditi and Albulak, 1986, p. 413), where Q_i, Q_j and t_i, t_j are the numbers and start times of the i^{th} and j^{th} units. Setting the finish time of the first unit ($Q1 = 1$) as t_1, Equation 2 returns the finish of the i^{th} unit in that chart.

$$m = (Q_j - Q_i) / (t_j - t_i) \quad \text{where } i < j \tag{1}$$

$$t_i = t_1 + (1/m) \cdot (Q_i - 1) \tag{2}$$

Activity Representation

In linear schedules, an activity that progresses over time is directly represented by a single line, whereas in LOB, it is *enveloped* by double lines (start and finish event). Having been established contemporaneously with network-based methods, the reason for such a fundamental difference may stem from LOB having been derived from activity-on-arrow (AOA) networks, as Lumsden (1968) describes at length, whereas LSM is rooted in the more recent activity-on-node (AON) representation. This distinction has been largely overlooked, despite some implicit evidence in the literature: Harris and Ioannou (1998, p. 270, *emphasis added*) applied AON to draw the CPM network for a single work unit before deriving a linear schedule, "because CPM diagrams show all of the *linkages between similar activities* in successive units, the number of links and *nodes* will likely be large and the network will appear unnecessarily complicated." Figures 1 and 2 illustrate the same activities A and B with their respective durations of 4 and

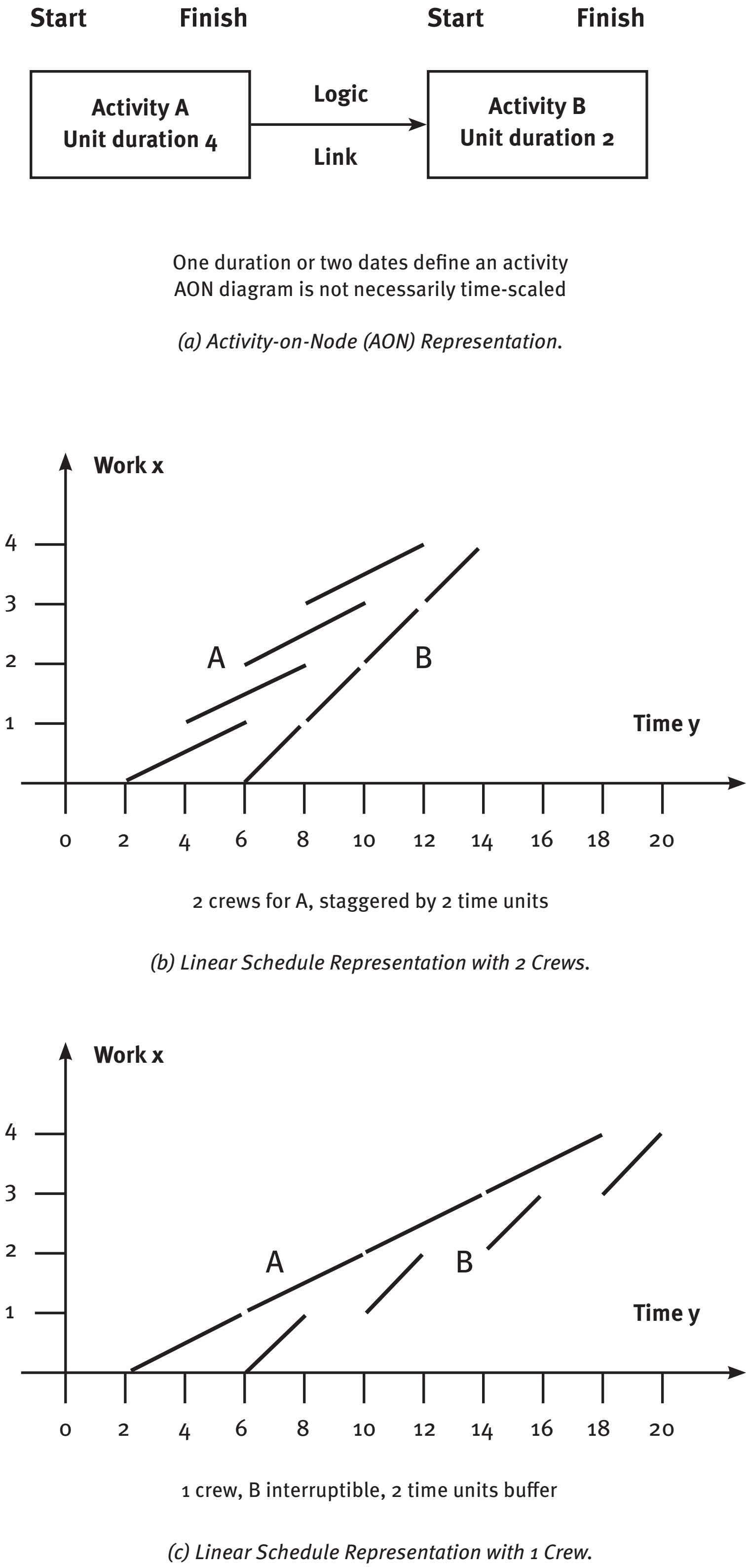


(a) Activity-on-Arrow (AOA) Representation

(b) LOB Quantity Representation with 2 Crews.

(c) LOB Quantity Representation with 1 Crew.

Figure 1: LOB Quantity is Generated from AOA (in part adapted from Lumsden, 1968, p. 5 / p. 15).



(a) Activity-on-Node (AON) Representation.

(b) Linear Schedule Representation with 2 Crews.

(c) Linear Schedule Representation with 1 Crew.

Figure 2: **Linear Schedule is Generated from AON (in part adapted from Harris and Ioannou, 1998, p. 271).**

2 time units per repetitive work unit. Figures 1a and 2a show the respective AOA and AON representation, where a circle is an event, which "unlike an activity, does not consume time or resources, it merely represents a point in time" (Lumsden, 1968, p. 5). The LOB representation of Figure 1b is significantly different from the linear schedule of Figure 2b. According to Ammar (2013), an activity in LOB forms a parallelogram within which each repeated instance is denoted by a horizontal bar. Different bars may be assigned to different crews. The width is equal with the duration of each unit activity, i.e. its length in a bar chart, and it ends "at the planned start and finish times of work in that unit" (Ammar, 2013, p. 45).

From this view, LOB can be considered to represent a combination of a traditional bar chart and a linear schedule. The slope of the finish event line represents the delivery rate of finishing repetitive units. Shifting the finish event line to the left by the unit duration returns the start event line. Together these two lines "describe the outer limits in time of our unit network which is repeating itself opposite each increment of the Line-of-Balance Quantity scale" (Lumsden, 1968, p. 15). Researchers thus implicitly used AOA when creating their networks for LOB quantity charts (Arditi *et al.*, 2002; Hegazy, 2001). Figures 1c and 2c show how changing the crew rate in the example from two crews to one crew results in modified progress slopes and a different pattern, which does not overlap anymore, but inserts interruptions into the progress.

Activity Start

In linear schedule diagrams, progress profiles of activities start at the origin, i.e. 0 units on the work axis, which is continuous. It cumulatively measures how much work has been completed after starting at nothing. Why, then, do profiles in LOB start growing from unit 1? The reason for this lies in the different

Characteristic	LSM	LOB
Slope means	Production rate	Delivery rate
General slope formula	Q / D	$(Q_j - Q_1) / (t_j - t_1)$
Time measuring	$D_j = t_{jF} - t_{1S}$, $D_1 = t_{1F} - t_{1S}$	$t_j - t_1$ implies finishes $t_{jF} - t_{1F}$
Quantity measuring	$Q_j = Q$ (if j is the last unit)	$Q_1 = 1$
Unified slope formula	Q_j / D_j Note $t_{1S} = 0$	$(Q_j - 1) / (D_j - D_1)$ Note $t_{jF} - t_{1F} = t_{jF} - t_{1S} - (t_{iF} - t_{1S}) = D_j - D_1$

Table 2: **Progress measurement differences of Linear Scheduling Method and Line-of-Balance**

meaning of slopes in LOB versus LSM: Slopes in linear schedules denote the *production rate*, but slopes in LOB are the *delivery rate* of finished units. Since "the Line-of-Balance method is geared to the delivery of completed units" (Lumsden, 1968, p. 14), the delivery rate only starts counting when the first unit has been finished. It thus becomes obvious that the LOB quantity axis in Figures 1b and 1c is not continuous like the LSM work axis in Figures 2b and 2c, but counts only integer work units. This is a fundamental difference between the two models. The discrete nature of LOB is a drawback, because it does not return a production quantity at non-integer times of interest, which would be important for monitoring and control.

Activity Productivity

In LSM, slope is directly proportional to the production rate, which equals the total quantity divided by the total duration. However, per Equation 1, the delivery rate in LOB is "indicating the speed by which work is to be finished in the repetitive units" (Hegazy, 2001, p. 125). It is called "natural rhythm" (Damci *et al.*, 2013, p. 683). Table 2 provides a detailed comparison of progress measurements between LSM and LOB: Regarding the slope, total quantity Q divided by the total duration D returns the productivity in linear schedules, but the quantity difference divided by the finish time difference returns the delivery rate in LOB. For measuring time, the total duration in LSM is the difference of the finish time of the last unit minus the start time of the first unit. But in LOB, the divisor is the difference of finish time of the last minus the finish time of the first unit itself. Analogously, for measuring quantity, the total quantity in LSM is the cumulative quantity that is finished at the last finish time. It is one when the first unit has been delivered. Comparing the unified formulas for slope of both LSM and LOB reveal that for the former it equals the cumulative quantity at the last unit divided by the passed duration (Q_j / D_j). But in LOB, one unit is subtracted from the count in the numerator and the duration of that unit in the denominator (Q_j – 1) / (D_j – D_1). This peculiar phenomenon can be explained via unified formulas: If only one crew performs all of the work continuously (or multiple crews work in a strict finish-to-start sequence, no overlapping), the slope in LSM is identical to the slope in LOB, because the *production rate* equals the *delivery rate*, as can be seen in Figures 1c and 2c. However, if multiple crews work concurrently with a lead time (i.e. a negative lag) in the start-to-start relation between crews, i.e. staggering, and each crew works continuously, then the slope of the single activity line in LSM will always be smaller than the slope of the two dashed event lines in LOB, as comparing the pattern of activity A in Figures 1b with 2b versus 1c with 2c shows.

To draw attention to this fundamental difference between both approaches and avoid confusion between a single line in Figures 2b and 2c, which directly tracks the actual productivity, and two enveloping diagonals in Figures 1b and 1c, they here are represented with dashed lines in a deviation from the traditional convention for LOB diagrams. Geometrically speaking, the line of LSM is a diagonal within each partial trapezoid of the LOB quantity chart, which directly connects the start event point of one work unit lower with the finish event point of the current work unit.

Of particular interest is also the manner in which crew assignments are visualized in both models. Staggering the crews is clearly represented by the bars in the LOB quantity charts. Note that the single crew of Figure 1c can work continuously on the four work units, as is shown by the dotted vertical steps between them. However, examining the crew that perform activity B shows that the strict requirement to maintain continuity within each work unit is causing a significant cost – that crew now must endure forced interruptions between each work unit of one time unit.

For multiple crews, the plan of Figure 1b shows that the overlap that is achieved by their staggered starts allow shortening the project duration from 20 to 14 time units. But this obviously comes at the cost of hiring an extra crew. The deliberate focus of

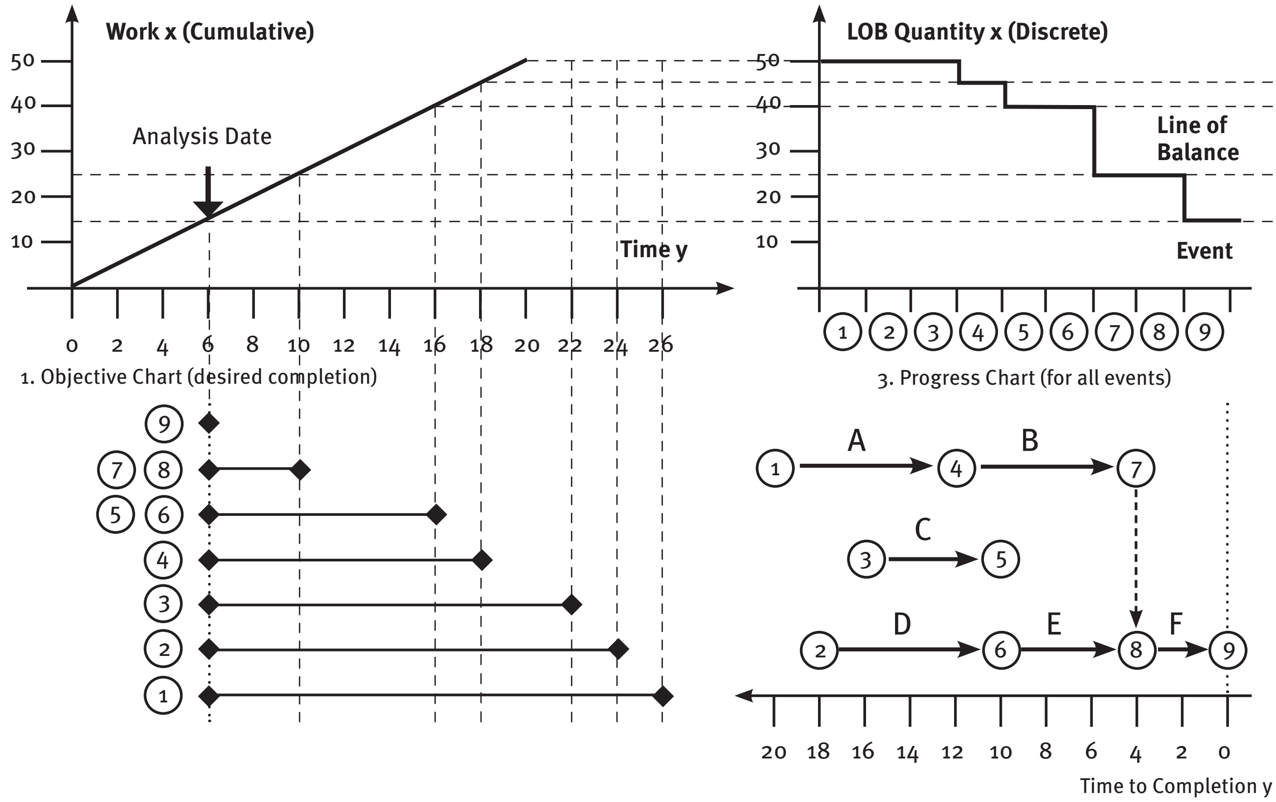


Figure 3: Steps of LOB in Manufacturing (based on Lumsden, 1968 and Office of Naval Material, 1962).

LOB on durations within each bar and double enveloping diagonals of delivery rate obscures these phenomena, whereas they are more clearly shown in LSM, which focuses on a continuous workflow (or in this case reveals the lack thereof). As can be seen from the figures for different crew assignments, unless durations of work units are fully aligned across activities, a need for either staggering or interruptability will inevitably arise. Reasons for fundamental differences in activity representation, start, and productivity between the LOB and LSM models have thus been analyzed and traced to their roots in AOA and AON, which fulfills Research Objective 1.

Manufacturing LOB Concepts

In lean manufacturing a counterpart to LOB exists, line balancing. It "is the process through which you evenly distribute the work elements within a value stream in order to meet takt time… it balances workloads so that no one is doing too little or too much" (Tapping *et al.*, 2002, p. 57). Note that the delivery rate of LOB is similar to the 'takt' of lean theory, whose German word means rhythm. "Takt time is the rate at which a company must produce a product to satisfy customer demand. Producing to takt means synchronizing the pace of production with the pace of scales" (Tapping *et al.*, 2002, p. 48). In comparison, LOB in manufacturing is "based on the principle of the assembly line balance… to meet the timing of the final assembly work" (Wang and Huang, 1998, p. 6). In its original application area, LOB has encompassed four elements: "THE OBJECTIVE – the cumulative delivery schedule. THE PROGRAM – the production plan. PROGRAM PROGRESS – the current status of performance. COMPARISON OF PROGRAM PROGRESS TO OBJECTIVE – the Line of Balance" (Office of Naval Material, 1962, p. 1, emphasis in original). For detailed illustration, the objective chart establishes the desired delivery schedule of the production process per the upper left part in Figure 3. Note that it resembles most closely a linear schedule in project management, rather than the aforementioned two-line enveloping LOB quantity chart. Then an 'assembly tree' per the lower right part of Figure 3 is established to serve as the detailed production plan. Its survey of "key plant operations or assembly points, and their lead-time relationship to final completion, is the most vital stage in a Line of Balance study" (ibid., p. 1). Note that this assembly tree resembles a bar chart schedule with logic links in project management, and its structure and content are "peculiar to the particular manufacturing process, from work on raw materials through assembly operations to point of shipment" (ibid., p. 2). Note also that this assembly tree continues to apply the AOA concept to model events as circles. Here it is time-scaled, whereby the length of the bar between them represents the duration and the distance between event markers represents lead or lag time for completing each stage. Note, importantly, that the time axis grows toward the left, because planning with LOB uses lead times before the contractually required delivery date. Next, the LOB analysis can be performed for any date of interest in the objective chart. Assume that the analysis date is at 6 time units as marked with an arrow. From this point on the time axis a vertical dotted line is drawn underneath the objective chart, from which multiple horizontal bars of the durations between the various events and the project completion (i.e. the final event) are traced from the adjacent production plan, here shown as bars with end markers. From the end markers, draw vertical dashed lines until they again intersect with the desired output line in the objective chart; then continue them to the right. The third and final chart is a column chart with the LOB quantity for each event. Events are simply assembled on the horizontal axis, which formally is a list and therefore does not feature an arrow tip. Again, this resembles a bar chart, albeit turned by 90 degrees, with the difference that the column heights are the required LOB quantities that cumulatively must have passed through each event (also called 'control point') to fulfill the desired output. Connecting the columns gives the name-giving stair-shaped Line-of-Balance, which is marked with a thick black line. It "specifies the quantities of end items sets for each control point which must be available in order for process on the program to remain in phase with the objective" (ibid., p. 5). In other words, the reason why the technique is called 'balance' is because this quantity graduation exactly fulfills the successor's demand in the assembly tree without accumulating any excess inventory to sustain a balanced production, no more, no less. LOB "is basically a tool for exercising surveillance over production programs" (ibid., p. 17), which bears a similarity to earned value management, but replaces tracking cost with counts.

This classic concept of LOB in manufacturing as is explained in Section 2.4 differs from the selected use of just the concept of LOB quantity charts that is used in construction project management, as has been explained in Section 2.1. The 'complete' LOB naturally has numerous advantages over the latter: First, it is sensible to call the resulting line in manufacturing LOB a 'balance', because it is a systematically derived measure of performance that fulfills the requirements at each event (control point) within the production system. Second, the classic manufacturing LOB has the potential to be expanded toward synergy with lean theory, because of its similarity with the line balancing in lean production. Third, the objective chart in Figure 3 is also linked with the linear schedule model, because both measure a quantity over time cumulatively and continuously. On the other hand, the production plan is essentially a bar chart with logic links over an inverse time axis. It is suggested that the 'partial' LOB concept that is currently applied in construction project management should better be called the 'multiple crews linear scheduling technique' if needed. It is a micro-level result that can be generated from the manufacturing LOB by plotting any adjacent two events in the AOA network from the assembly tree to which a vertical LOB quantity axis is added, and a progress slope that is given by how many crews are employed in a staggered manner. It may thus be considered a side-product that has evolved out of LOB as it was originally intended and used. On the other hand, the manufacturing LOB is plotting the complete AOA network, so it provides a more general functionality by covering both the micro-level and macro-level of project planning. Having examined the conceptual relationships and strengths of the two different yet related LOB concepts of manufacturing and construction management fulfills Research Objective 2.

LOB MODEL WITH SINGULARITY FUNCTIONS

After reviewing and noting the fundamental differences between LSM and LOB and within LOB concepts, it is possible to develop a mathematical model of LOB based on existing LSM equations, which will aid in unifying these two methods. Two possible ways exist to establish such LOB mathematical expression: 1. Model the dashed event lines "which describe the outer limits in time of our unit network" (Lumsden, 1968, p. 15); or 2. model the horizontal bar chart for each crew within each activity. For the former, one must model the start event line, the finish event line then will be generated via adding the duration of each crew. But if different crews have different durations, or dissimilar lag times, then this would increase the complexity of such model. For the latter, a bar chart profile with singularity functions has been realized successfully (Lucko and Su, 2014). Therefore this paper will create the LOB equations by extending the previous bar chart concept. A model that is derived from the event line concept will be covered in future research.

Singularity Functions

Singularity functions originate in structural engineering to express different types of loads along a member. They are piecewise functions with jump (or bend) discontinuities (the eponymous singularities). Equation 3 is the basic case distinction that composes all singularity functions. By switching the equal sign from the lower to the upper case, one could redefine it from right- to left-continuous. If the independent variable x is within the upper domain (x ‹ a), the term is equal to zero. Else, the term activates and treats the pointed bracket operator as round brackets of normal algebra. Parameters are the scale factor s, cutoff value a, and exponent n, which jointly determine how the singularity function will behave: A step function (for *n* = 0, s is a step height, a is the activation location); a linear function (for *n* = 1, s is a slope that activates at a); or a nonlinear function (for *n* › 1, s defines the growth intensity).

$$s \cdot \langle x-a \rangle^n = \begin{cases} 0 & \text{for } x < a \\ s \cdot (x-a)^n & \text{for } x \geq a \end{cases} \quad (3)$$

LOB and LSM Equations

The LOB and LSM equations of each crew are provided by Equation 4 and 5, respectively. They express the performance of one crew on one work unit (i.e. a task). The aS and aF are the start and finish time of said crew's task. In Equation 4, x_S denotes the start unit of the work. And v_C assigns how many units each crew must complete per cycle, which need not necessarily be an integer value (as LOB has traditionally assumed). It is the height between steps on the finished unit axis of LOB. Whereas the slope of LSM is the productivity of one crew in work units per time units. As a result, the finish time equals the start time plus the duration, which is the ratio of v_C and the slope itself. In Equation 5, the term with exponent zero defines the intercept of the LSM profile on the work axis (e.g. at which work unit the task starts), which allows modeling the task for any unit of work. The analogous LOB model thus adds the intercept x_S into the factor before the on and off switch terms. Substituting the data of the example into the model parameters, Table 3 lists the respective LSM and LOB equations for the scenarios of Figures 1b and 2b.

Note that in Equations 4 and 5 the duration of each task i is given by a term that relies upon the slope in LSM. Alternatively, the task duration d may be known explicitly and could then be inserted directly in the LOB equation.

$$x(y)_{LoB} = (x_S + v_C) \cdot (\langle y-a_S \rangle^0 - \langle y-a_F \rangle^0)$$

$$\text{where } a_F = a_S + \frac{v_C}{slope_{LSM}} = a_S + d \quad (4)$$

$$x(y)_{LSM} = x_S \cdot \langle y-a_S \rangle^0 + slope_{LSM} \cdot (\langle y-a_S \rangle^1 - \langle y-a_F \rangle^1) \text{ where}$$

$$a_F = a_S + \frac{v_C}{slope_{LSM}} \quad (5)$$

Characteristics of LOB and LSM Models

Various scheduling scenarios can be modeled by these LOB and LSM models, which includes the staggered (i.e. concurrent crews within the same activity), continuous, and interruptible scenarios per Figures 4 and 5. The user can customize the slope, start time, start unit, and number of units assigned to each crew for each activity as needed. For brevity, detailed equations for each scenario are omitted here, but can be created by following Section 3.2.

The parameter values are listed in the title of each figure. Note that the time buffer between A and B is zero. As Figures 4a to 4c and 5a to 5c show, by modeling LSM at the crew level, LOB and LSM can be converted into one another directly, and the traditional LSM at the activity level is just the specific case of Figure 5b. All of the three scenarios – staggered, continuous, and interruptible – are generated by setting the lag time to be negative (i.e. a lead), zero, or positive. However, a shortcoming of LOB is apparent in Figure 4d: If a crew does not start from the first work unit, there may exist some uncertainty about its initial performance, which the more explicit representation of

Activity	Work Unit	LSM equations	LOB equations
A	1	$x_{LSM\_A\_1} = 0 \cdot \langle y-2 \rangle^0 + 1/4 \cdot (\langle y-2 \rangle^1 - \langle y-6 \rangle^1)$	$x_{LOB\_A\_1} = 1 \cdot (\langle y-2 \rangle^0 - \langle y-6 \rangle^0)$
A	2	$x_{LSM\_A\_2} = 0 \cdot \langle y-4 \rangle^0 + 1/4 \cdot (\langle y-4 \rangle^1 - \langle y-8 \rangle^1)$	$x_{LOB\_A\_2} = 2 \cdot (\langle y-4 \rangle^0 - \langle y-8 \rangle^0)$
A	3	$x_{LSM\_A\_3} = 0 \cdot \langle y-6 \rangle^0 + 1/4 \cdot (\langle y-6 \rangle^1 - \langle y-10 \rangle^1)$	$x_{LOB\_A\_3} = 3 \cdot (\langle y-6 \rangle^0 - \langle y-10 \rangle^0)$
A	4	$x_{LSM\_A\_4} = 0 \cdot \langle y-8 \rangle^0 + 1/4 \cdot (\langle y-8 \rangle^1 - \langle y-12 \rangle^1)$	$x_{LOB\_A\_4} = 4 \cdot (\langle y-8 \rangle^0 - \langle y-12 \rangle^0)$
B	1	$x_{LSM\_B\_1} = 0 \cdot \langle y-6 \rangle^0 + 1/2 \cdot (\langle y-6 \rangle^1 - \langle y-8 \rangle^1)$	$x_{LOB\_B\_1} = 1 \cdot (\langle y-6 \rangle^0 - \langle y-8 \rangle^0)$
B	2	$x_{LSM\_B\_2} = 0 \cdot \langle y-8 \rangle^0 + 1/2 \cdot (\langle y-8 \rangle^1 - \langle y-10 \rangle^1)$	$x_{LOB\_B\_2} = 2 \cdot (\langle y-8 \rangle^0 - \langle y-10 \rangle^0)$
B	3	$x_{LSM\_B\_3} = 0 \cdot \langle y-10 \rangle^0 + 1/2 \cdot (\langle y-10 \rangle^1 - \langle y-12 \rangle^1)$	$x_{LOB\_B\_3} = 3 \cdot (\langle y-10 \rangle^0 - \langle y-12 \rangle^0)$
B	4	$x_{LSM\_B\_4} = 0 \cdot \langle y-12 \rangle^0 + 1/2 \cdot (\langle y-12 \rangle^1 - \langle y-14 \rangle^1)$	$x_{LOB\_B\_4} = 4 \cdot (\langle y-12 \rangle^0 - \langle y-14 \rangle^0)$

Table 3: **Equations for Figures 1b and 2b**

LSM does not encounter. For example, Figure 4b could be alternatively explained as crew 1 of A finishing 3.5 units if it starts at 0, or 1 unit if starting at 2.5, or 1.5 units if starting at 2, andsoforth. The reason for that is the graduation from the vertical work axis in LOB measures finished units, not just units as in LSM. Users may chose to use LOB, assuming that activities start at 1, i.e. finishing the first work unit, but will lose some exact information about the exact start.

Another shortcoming is the fact that overlapping arrows may occur in LOB, e.g. for tasks A4 and B4 in Figures 4b and 4c. On the other hand, the LSM representation in Figures 5b and 5c more clearly shows the brief period where these two tasks are concurrent within the same work unit, which is easily identified by the end points of both lines touching. Overall, the new singularity functions allow modeling all scenarios, which fulfills Research Objective 3.

RESOURCES WITH SINGULARITY FUNCTIONS

An intriguing opportunity arises from the fact that singularity functions can model specific work units within an activity as the previous section has described. Since both LOB and LSM describe repetitive tasks, i.e. specific crews performing specific work units, the modeling effort should focus on how this important relation can be expressed in detail. This leads to the question of how the required number of crews relates to staggering. The following inequality conditions are derived from the examples of Figures 4 and 5 for any activity with segments of constant productivity:

- If *crew* › 1 and *lag time* ‹ task duration (LOB) or *lag time* ‹ v_c / *LSM slope* (LSM), then resource staggering occurs. An optimum lag time can be determined so that crews incur zero idle time when transitioning to their next task. Such optimum would employ the number of crews as a fraction of the *task duration* to space the offset between tasks evenly: *Lag time = task duration / crews* (LOB) or *lag time* = (v_c / *LSM slope*) / *crews* (LSM);
- If *crew* = 1 (multiple resources would be discontinuous) and *lag time = task duration* (LOB) or *lag time* = v_c / *LSM slope* (LSM), then resource continuity is maintained among tasks (i.e. LSM segments) within an activity;
- If *crew* ≤ 1 and *lead time* › [*task duration* or v_c / *LSM slope*] (LOB or LSM), then resource interruption occurs.

Of course, at least one crew must exist for any of these scenarios to occur. Specific to the staggering scenario, the number of required crews can be calculated as [*task duration* or v_c / *LSM slope*] / *lag time*, rounded up to an integer. The idle time for the interruption scenario would be a *lag time*.

Crew-Task Equations

Specific equations that capture each crew-task combination individually can theoretically be derived by using the task number i as a switch operator within the model to generate different tasks or segments as output. Equations 6 and 7 are the LOB and LSM crew-task equations for a given task i based on a known start a_{s_o} of the entire activity and assuming resource continuity can be achieved. They determine the respective start and finish of the singularity function with multiples of i that offset the task by several lag times to achieve the desired cumulative staggering. In the case of the aforementioned continuous scenario for one *crew*, the *lag time* is equal to the *task duration* d itself. Of course, it is then easy to extract exactly only those repeated tasks that a crew performs as it traverses through the activity: The valid values of i increases by multiples of the total number of crews. For example, *crew* i = 2 among 5 total crews would work on the tasks that are numbered 2, 2 + 5 = 7, 2 + 5 + 5 + 12, andsoforth and skip other tasks.

$$x(y)_{LOB\_i} = i \cdot v_c \cdot (\langle y-(a_{s\_o}+ (i - 1 \cdot \mathit{lag\ time}) \rangle^{0} - \langle y-(a_{s\_o}+(i - 1) \cdot \mathit{lag\ time}) + d \rangle^{0}) \quad (6)$$

$$x(y)_{LSM\_i} = (i - 1) \cdot v_c \cdot (\langle y-(a_{s\_o} +(i - 1) \cdot \mathit{lag\ time}) \rangle^{0} + \mathit{slope}_{LSM} \cdot (\langle y-(a_{s\_o}+(i - 1) \cdot \mathit{lag\ time}) \rangle^{1} - \langle y-(a_{s\_o}+(i - 1) \cdot \mathit{lag\ time}) + d \rangle^{1}) \quad (7)$$

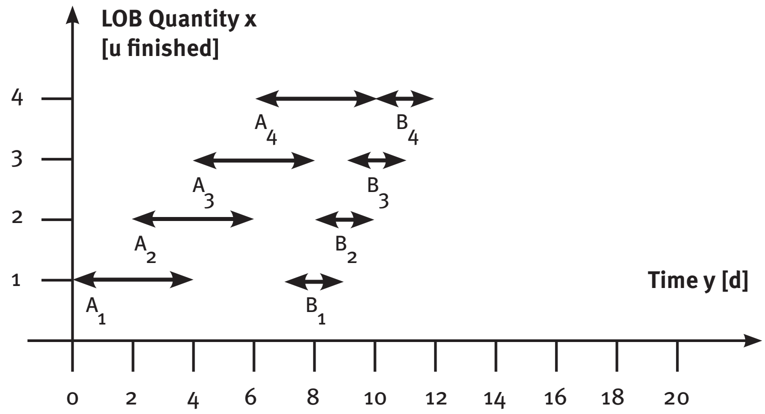
Time and Work Buffers

Maintaining buffers between the segments at this resource level of detail can be enforced. For time buffers, this requires that the tasks from two adjacent (i.e. predecessor-successor) activities for the same work unit maintain sufficient horizontal distance in Figures 4 and 5. Rather than checking such constraint point-wise between each pair of tasks graphically, it would be easier to insert it as an additive term after the '+ d' in the LOB and LSM equations. An equivalent work buffers could be realized by converting it into a time buffer via the *LSM slope* (this would add the work buffer after the predecessor) and positioning the successor so that it coincides with either the finish plus buffer of the first or last task, depending on their relative slopes. For converging productivities (where *LSM slope* of predecessor ‹ *LSM slope* of successor), a finish-to-finish relation emerges between the two activities and their last tasks come in closest proximity. For diverging productivities (*LSM slope* of predecessor › *LSM slope* of successor), a start-to-start relation emerges and their first tasks are closest. Another way to model a work buffer is checking step heights of the adjacent activity and adding it into the factor at the beginning of Equations 6 and 7 whenever needed.

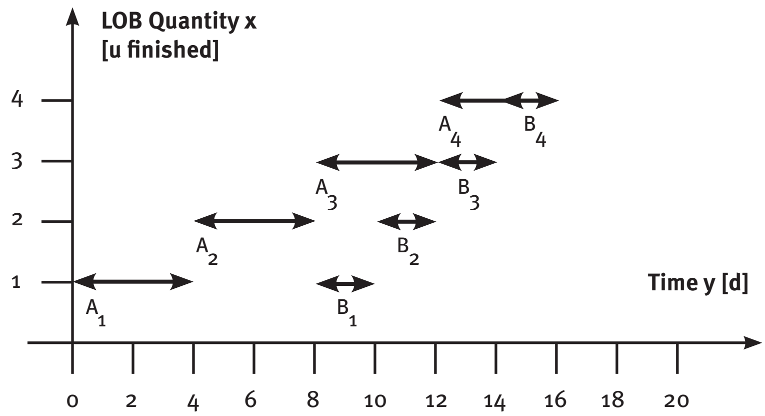
Resource Histogram

Having created crew-task equations at a detailed level, it is possible to derive a resource histogram. Lumsden (1968) already considered a resource histogram to be an integral part of the LOB analysis and in fact presented it not just as a separate graphic, but also overlaid to the LOB diagram itself. But such a graphical approach to representing resource use over time throughout the LOB literature (and also studies that used network schedules or bar charts) has been an obstacle in terms of modeling versatility. Therefore, it will be explored how a mathematical expression for the resource histogram can be derived from the previously established crew-task equations. Conceptually speaking, two (or three steps) are needed. First, converting the crew-task equation into a resource equation for that crew and task. Second, adding them within an activity. Third, adding them across activities within the project, if it is desired.

To accomplish the transformation from a crew that works on a specific task to a generic crew that can be added, the task number i is replaced by a resource count r_i in Equation 8. For LSM it also modified the exponent from n = 1 (slope) to n = 0 (step). It typically is $r_i = 1$ to create a resource histogram at the crew level, or could be the number of laborers within said crew. A sum of resource equations is a superposition (Lucko 2008), which should be simplified for Equation 9 by adding the steps s all of the basic terms of the form of Equation 3 that have the same cutoff a (i.e. start time) and exponent n (here zero). This would give the shortest mathematical expression for a specific resource histogram. Or start and finish variables could be kept, which gives a general expression for all resource histograms. Figure 6 shows the resource histogram for the scenario of Figures 4/5a through 4/5d. Note that they contain only one or two crews. In the latter case, they may alternate, which is reflected in the total resource count on the vertical axis.



(a) LOB staggered with 2 crews (A: y_S = 0 d, x_S = 0 u, v_C = 1 u/crew, lag time = -2 d (lead), task duration = 4 d, delivery rate = 1/2 u/d, time buffer = 0 d; B: y_S = 7 d, x_S = 0 u, v_C = 1 u/crew, lag time = -1 d (lead), task duration = 2 d, delivery rate = 1 u/d).



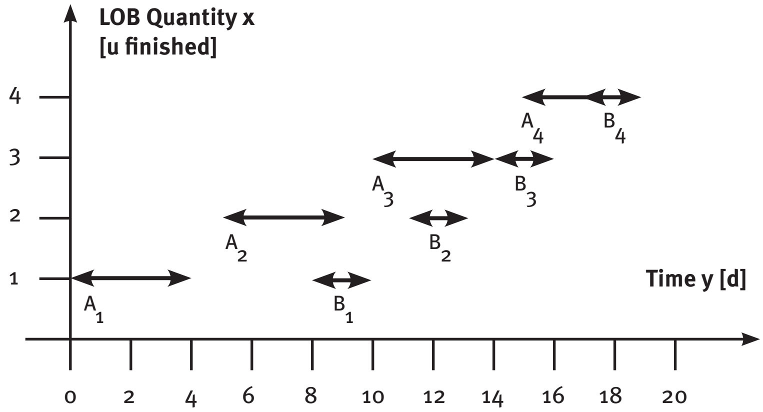
(b) LOB continuous with 1 crew (A: y_S = 0 d, x_S = 0 u, v_C = 1 u/crew, task duration = 4 d, delivery rate = 1/4 u/d, time buffer = 0 d; B: y_S = 8 d, x_S = 0 u, v_C = 1 u/crew, task duration = 2 d, delivery rate = 1/2 u/d).

Figure 4: Four Scenarios in LOB

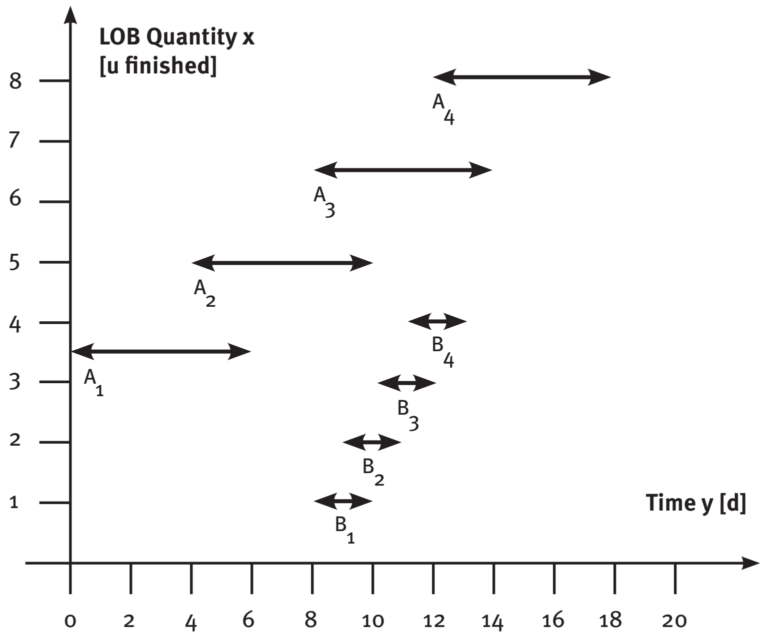
$$r(y)_{res\_i} = r_i \cdot (\langle y-(a_{s\_o}+(i-1)\cdot \textit{lag time})\rangle^0 - \langle y-(a_{s\_o}+(i-1)\cdot \textit{lag time} + d\rangle^0) \quad (8)$$

$$r(y)_{res\_histogram} = \Sigma\, x(y)_{res\_i} \quad (9)$$

This approach has transformed individual resource equations into resource counts, which are added from a start to a finish date to give an activity histogram. They can be added further along the time axis to give the entire project histogram. A more direct approach is possible based on the characteristic trapezoidal shape of a resource histogram. It contains four segments per Figure 7 from earlier to later on the time axis and beginning at an activity start.



(c) LOB interruptible with 1 crew (A: $y_s = 0$ d, $x_s = 0$ u, $v_c = 1$ u/crew, lag time = 1 d, task duration = 4 d, delivery rate = 1/5 u/d, time buffer = 0 d; B: $y_s = 8$ d, $x_s = 0$ u, $v_c = 1$ u/crew, lag time = 1 d, task duration = 2 d, delivery rate = 1/3 u/d).



(d) LOB staggered with 2 crews (A: $y_s = 0$ d, $x_s = 2$ u, $v_c = 1.5$ u/crew, lag time = -2 d (lead), task duration = 4 d, delivery rate = 3/8 u/d, time buffer = 0 d; B: $y_s = 8$ d, $x_s = 0$ u, $v_c = 1$ u/crew, lag time = -1 d (lead), task duration = 2 d, delivery rate = 1 u/d).

These segments are an upslope, a plateau (if any), a downslope (which due to assuming the activity to have the same crew-task durations is the same as the upslope), and zero resources in the open interval after said activity. It is necessary to convert what would be a straight-segment trapezoid into the stepped shape. This is accomplished by introducing the roundup operator into the singularity function of Equation 10. This equation can calculate the resource trapezoidal shape by knowing the overall *activity start* (a_S), *finish* (a_F), *delivery rate* (v_d), and *peak*, where the slope of the dashed lines in the resource histogram of Figure 7 is the delivery rate (v_d in Equation 10) from Table 2.

Conclusions and recommendations

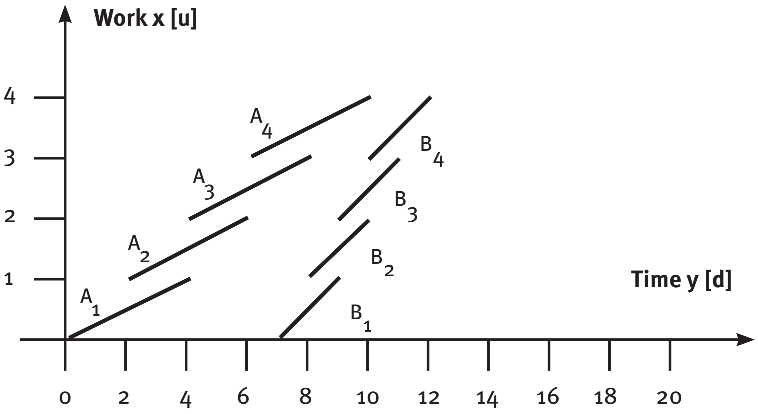
LOB is a unique resource-driven scheduling technique that holds significant potential for beneficial application to construction projects with repetitive activities. However, several of its basic characteristics appeared to mismatch those of linear schedule models. Driven by its stated motivation to compare and contrast LOB and LSM and identify differences and commonalities, this paper has thus systematically reviewed their concepts with regards to the major aspects of activity representation, start, and productivity. Its findings contribute to the body of knowledge in several ways: First, LOB and LSM are found to have been conceptually based on AOA and AON representations of network schedules, which explains why double lines envelope activities in LOB, whereas linear schedule represent them with a single line. Second, the reason that the LOB quantity chart starts at 1 work unit is that the slope in LOB describes the delivery rate in integer increments. Since the delivery rate counts how fast

$$r(y)_{res\_hist} = \left(\left\langle \left\lceil \frac{y - a_S}{v_d} \right\rceil - 0 \right\rangle^1 - \left\langle \left\lceil \frac{y - (a_S + peak \cdot v_d)}{v_d} \right\rceil - 0 \right\rangle^1 \right) - \left(\left\langle \left\lceil \frac{y - (a_F - (peak - 1) \cdot v_d)}{v_d} \right\rceil - 0 \right\rangle^1 - \left\langle \left\lceil \frac{y - a_F - v_d}{v_d} \right\rceil - 0 \right\rangle^1 \right) \text{ where } v_d = \frac{Q_j - Q_1}{t_j - t_1} \quad (10)$$

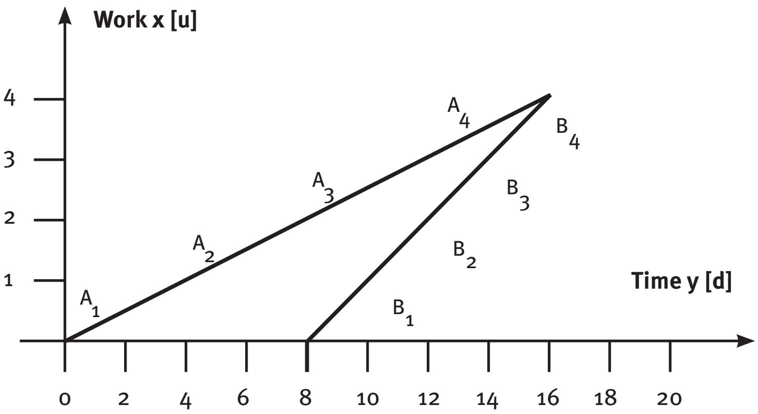
work units are finished, it starts at the finish of the first unit. Furthermore, progress measurements in LOB and LSM have been explained mathematically and graphically. Third, the full manufacturing LOB exceeds the analytical capabilities of LOB in the form that has been used in construction, yet also has potential synergy with lean production theory. Finally, since LSM has already been successfully expressed with singularity functions in an accurate formulation (Lucko, 2008), their mathematical model is extended LSM to newly providing LOB, including staggering, continuous, and interruptible scheduling scenarios.

Recommendations for future research include that the model should be extended further to explore activities that have variable production rates between tasks or planned interruptions on the time axis or gaps on the work axis. A comprehensive conversion algorithm should be established to transform LSM into LOB schedules and vice versa, so that both methods can be used in synchrony by users. The conceptual connections of these two related methods with network schedules of the critical path method as well as Gantt bar charts should also be explored in more depth, now that conceptual roots of LOB and LSM in AOA and AON have been explicitly revealed. While the former methods are two-dimensional – they comprise time and work – the latter are essentially one-dimensional, but it would still be worthwhile to formalize the dimensional step in information content between them with formulas and an algorithm.

Considering the resource-driven and productivity-focused nature of LOB and LSM, respectively, an opportunity presents itself to explore resource-related phenomena in more depth. For example, the model could be expanded to handle multiple different types of resources. It should also be investigates how more flexible resource use akin to job shop scheduling could be accommodated. The newly derived singularity functions for LOB and LSM themselves are somewhat limited in that they describe individual segments for each tasks within an activity. It would streamline the model if the segmented equations could be even further integrated, e.g. using a task operator. Equipped with such a flexible mathematical framework, a comprehensive unification of the various scheduling techniques appears possible.

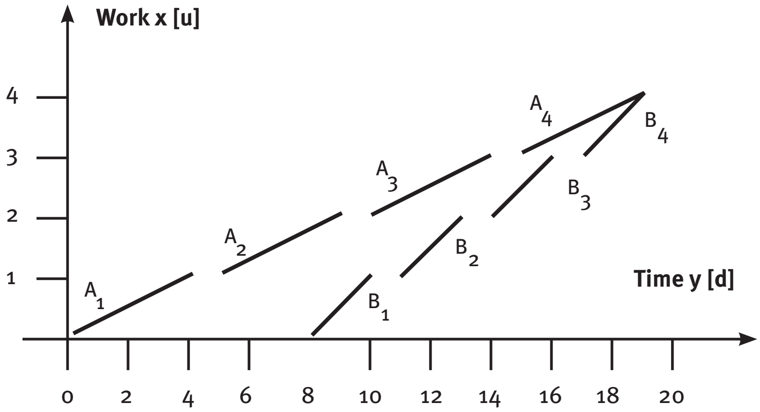


(a) LSM staggered with 2 crews (A: y_S = 0 d, x_S = 0 u, v_C = 1 u/crew, lag time = -2 d (lead), slope = 1/4 u/d, time buffer = 0 d; B: y_S = 7 d, x_S = 0 u, v_C = 1 u/crew, lag time = -1 d (lead), slope = 1/2 u/d).

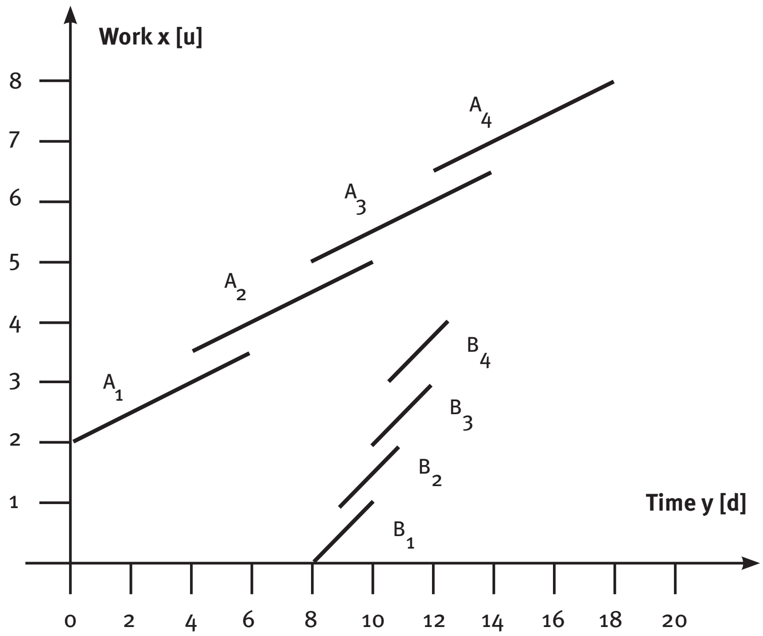


(b) LSM continuous with 1 crew (A: y_S = 0 d, x_S = 0 u, v_C = 1 u/crew, slope = 1/4 u/d, time buffer = 0 d; B: y_S = 8 d, x_S = 0 u, v_C = 1 u/crew, slope = 1/2 u/d).

Figure 5: **Four Scenarios in LSM**



(c) LSM interruptible with 1 crew (A: y_S = 0 d, x_S = 0 u, v_C = 1 u/crew, lag time = 1 d, slope = 1/4 u/d, time buffer = 0 d; B: y_S = 8 d, x_S = 0 u, v_C = 1 u/crew, lag time = 1 d, slope = 1/2 u/d).



(d) LSM staggered with 2 crews (A: y_S = 0 d, x_S = 2 u, v_C = 1.5 u/crew, lag time = -2 d (lead), slope = 1/4 u/d, time buffer = 0 d; B: y_S = 8 d, x_S = 0 u, v_C = 1 u/crew, lag time = -1 d (lead), slope = 1/2 u/d).

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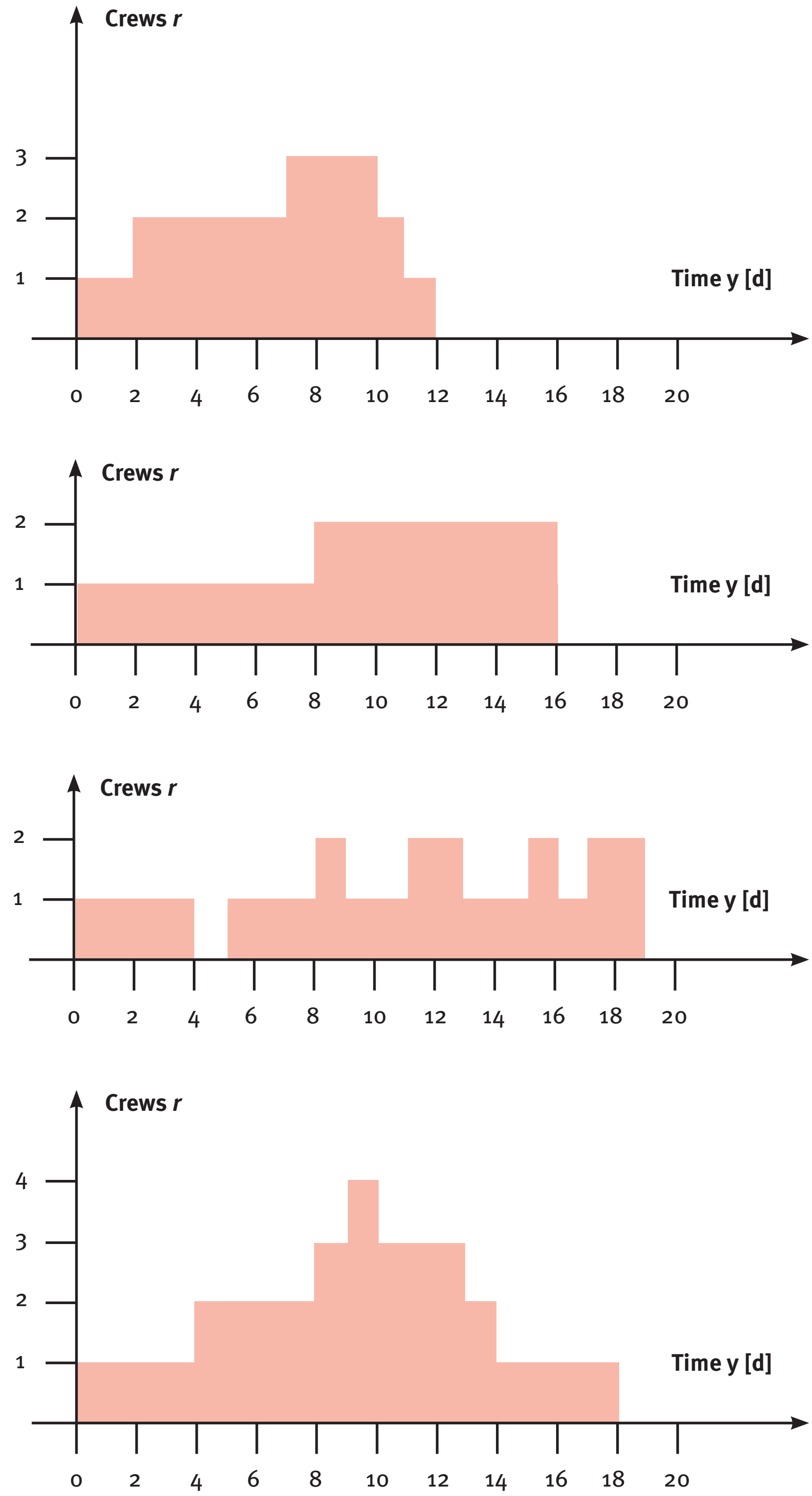

Figure 6: **Resource Histogram for LOB and LSM Examples.**

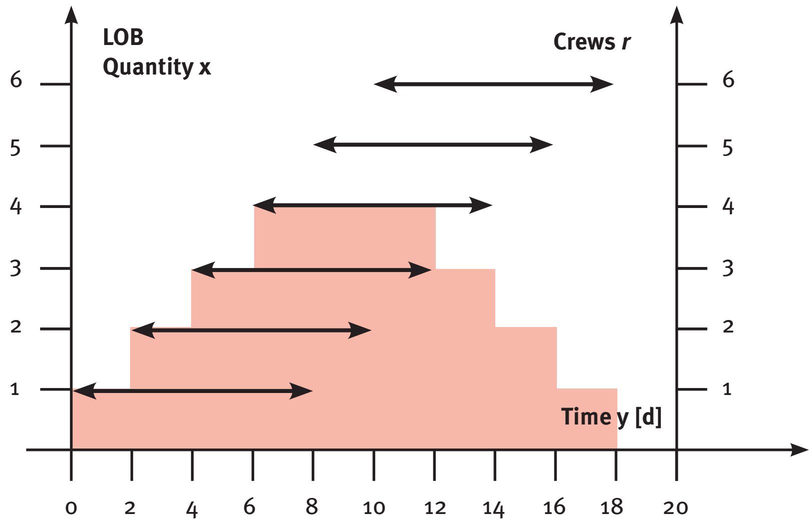

Figure 7: Resource Histogram for Generic LOB Activity.

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